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Well-posedness for Optimization Problems with Constraints defined by Variational Inequalities having a unique solution

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Abstract. We introduce various notions of well-posedness for a family of variational inequalities and for an optimization problem with constraints defined by variational inequalities having a unique solution. Then, we give sufficient conditions for well-posedness of these problems and we present an application to an exact penalty method.

Key words: Optimization problems with variational inequalities constraints, Bilevel programming problems, Stackelberg problems, Approximating sequences, Strong well-posedness, Generalized well-posedness, Gap function, Monotone operators

1. Introduction

Let X be a topological space and E be a reflexive Banach space with dual E^* and K be a nonempty convex closed subset of E. For a function f from $X \times E$ to $\mathbb{R}$ we consider the following Optimization Problem with Variational Inequality Constraints, denoted by OPVIC, also called generalized bilevel programming with Equilibrium Constraints by Luo et al. [10]:

$$\text{(OPVIC)} \qquad \begin{cases} \text{Minimize } f(x, u) \\ \text{subject to } (x, u) \in X \times E \text{ and } u \in T(x). \end{cases}$$

$T(x)$ is the solution set of the parametric variational inequality (VI)(x) defined by the pair $(A(x, .), K)$, $A(x, .)$ being an operator from E to E^*, i.e. $u \in T(x)$ if and only if $u \in K$ and satisfies the inequality:

$$\langle A(x, u), u - v\rangle \leqslant 0 \quad \forall v \in K.$$

Problems of this type have been investigated by many authors (Marcotte and Zhu [11]; Outrata [14]; Ye et al., [16]; Luo et al. [10]; Lignola and Morgan [9]) mostly in the setting of finite dimensional spaces. Aim of this paper is to introduce a concept of well-posedness for OPVIC which can be useful for numerical purposes. More precisely, in line with the concepts of well-posedness given in Optimization (Tychonov [15]; Dontchev and Zolezzi [4]); in Game Theory (Cavazzuti and Morgan [3]; Margiocco et al. [12]) and in Bilevel Optimization (Morgan [13]), we introduce the notion of 'approximating sequences' for OPVIC. Then, we determine classes of problems which guarantee the strong convergence of such sequences to a solution of the original problem. So any algorithm or method which produces 'approximating sequences' allows us to approach a solution. We shall see in Section 3 that there is a strong connection between well-posedness for OPVIC and 'parametric well-posedness' of the family of variational inequalities $\{VI(x), x \in X\}$ (see Definition 2.2). Thus, in Section 2 we define parametrically well-posed families of variational inequalities, we give classes of operators ensuring such a property and we investigate the relationship with other possible concepts of parametric well-posedness. Then in Section 3 we give sufficient condition for well-posedness of OPVIC and we present an application to an exact penalty method for OPVIC (Example 3.4). For all the definitions related to Variational Inequalities (monotonicity, hemicontinuity, ...), we refer to Baiocchi and Capelo [2]).

We point out that, in the following, we deal only with variational inequalities having a unique solution; the case of a non unique solution will be considered separately.

2. Parametrically Well-posed Variational Inequalities

In this section, for $x \in X$, we consider the parametric variational inequality:

$$(VI(x)) \begin{cases} \text{Find } u \in K \text{ such that:} \\ \langle A(x,u), u - v\rangle \leqslant 0 \quad \forall v \in K, \end{cases}$$

and we assume that, for all $x \in X$, $(VI)(x)$ has a unique solution.

As observed by Harker and Pang [7] and by Marcotte and Zhu [11], the problem $(VI)(x)$ can be reformulated in the following way:

$$P(x) \begin{cases} \text{find } u \in K \text{ such that :} \\ g(x,u) = 0 \text{ and } \inf_{v\in K} g(x,v) = g(x,u), \end{cases}$$

where

$$g(x,v) = \operatorname*{Sup}_{w\in K}\langle A(x,v), v - w\rangle. \tag{1}$$

The 'gap' function g, introduced by Auslender [1], which provides an optimization formulation for the problem $(VI)(x)$, is used in various numerical methods for solving variational inequalities (see, for example, Harker and Pang [7]; Marcotte and Zhu [11]). Thus, assuming that the problem $(VI)(x)$ has at least a solution, it is natural to give the following definitions:

DEFINITION 2.1. Let $x \in X$ and (x_n) be a sequence converging to x. A sequence (u_n) is an *approximating sequence* for the problem $(VI)(x)$ (with respect to (x_n)) if $u_n \in K$ for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} g(x_n, u_n) = 0$, that is there exists a sequence of positive numbers (ε_n) converging to zero such that:

$$\langle A(x_n, u_n), u_n - v\rangle \leqslant \varepsilon_n \quad \forall v \in K.$$

Now, let us consider the family $(VI) = \{(VI)(x), x \in X\}$.

DEFINITION 2.2. The family (VI) is parametrically strongly well-posed if:

(i) there exists a unique solution $\bar{u}_x$ to $(VI)(x)$, for all $x \in X$;

(ii) for all $x \in X$ and for all (x_n) converging to x, every approximating sequence for the problem $(VI)(x)$ (w.r. to (x_n)) strongly converges to $\bar{u}_x$.

Now, we investigate the connection between the concept of parametric well-posedness given by Definition 2.2 and the diameter of the set

$$T(x, \varepsilon) = \{u \in K : \langle A(x, u), u - v\rangle \leqslant \varepsilon \quad \forall v \in K\} \text{ for } \varepsilon \geqslant 0$$

as defined in Lignola and Morgan [8] in which continuity properties have been studied. Unfortunately, differently from what happens in Optimization (Dontchev and Zolezzi [4]), in general parametrically well- posedness is not equivalent to the convergence of the diameters of $T(x, \varepsilon)$ to 0. In fact only one implication holds and more precisely:

PROPOSITION 2.3. *If the family (VI) is parametrically strongly well-posed then $T(x, \varepsilon) \neq \emptyset$, for every $\varepsilon > 0$ and every $x \in X$, and $\lim_{n\to\infty}$ diam $T(x_n, \varepsilon_n) = 0$, for all (x_n) converging to x and all (ε_n) converging to 0.*

Proof. Let (VI) be parametrically strongly well-posed and g be the gap function as defined in (1). Assume that there exist $x \in X$, (x_n) converging to x and (ε_n) converging to 0 such that $\lim_{n\to\infty} \text{diam} T(x_n, \varepsilon_n) > 0$. Then one can find a positive number a and two sequences in $T(x_n, \varepsilon_n)$, (u_n) and (v_n), such that

$$\|u_n - v_n\| > a \text{ for all } n. \tag{2}$$

Being (u_n) and (v_n) two approximating sequences for the problem $(VI)(x)$ (w.r. to (x_n)), they have to converge to the unique solution for $(VI)(x)$ in contradiction with (2). □

When the operator A does not depend on the parameter x we obtain a stronger result:

PROPOSITION 2.3 bis. *Let A be a monotone and hemicontinuous operator from E to E^*. Then (VI) is well-posed if and only if $T(\varepsilon) \neq \emptyset$ for any $\varepsilon > 0$ and $\lim_{\varepsilon\to 0}$ diam $T(\varepsilon) = 0$.*

Proof. The following Lemma is well known (Kinderlehrer and Stampachia [6]; Baiocchi and Capelo [2]), however, we shall give its proof for sake of completeness.

MINTY'S LEMMA. *Let A be a map from E to E^* and K be a subset of E. If A is monotone, then any solution of the variational inequality*

$$\langle Au, u - v\rangle \leqslant 0 \quad \forall v \in K.$$

is also a solution of the inequality

$$\langle Av, u - v\rangle \leqslant 0 \quad \forall v \in K. \tag{3}$$

Conversely, if A is hemicontinuous and K is convex then any solution u in K of (3) is also a solution to (VI).

Proof. The first part of the Lemma is a trivial consequence of the monotonicity of A. Conversely, let $u \in K$ be a solution of (3) and v be an arbitrary vector of K. The vector

$$v_t = tv + (1 - t)u, 0 < t < 1,$$

belongs to K for all t, since K is convex. Hence, by (3),

$$\langle Av_t, u - v_t\rangle \leqslant 0,$$

that is to say,

$$\langle Av_t, u - v\rangle \leqslant 0.$$

Therefore, if t converges to 0, we find by the hemicontinuity of A,

$$\langle Au, u - v\rangle \leqslant 0.$$

Then u satisfies (VI). □

Proof. Let (u_n) be an approximating sequence for (VI), that is:

$$u_n \in K \text{ and } \langle Au_n, u_n - v\rangle \leqslant \varepsilon_n \quad \forall v \in K$$

and $\varepsilon_n \to 0$.

If $\lim_{\varepsilon\to 0} \operatorname{diam} T(\varepsilon) = 0$, for all $\eta > 0$ there exists $m \in \mathbb{N}$ such that: diam $T(\varepsilon_n) < \eta$ for all $n \geqslant m$. Hence $\|u_p - u_q\| < \eta$ for all $p, q \geqslant m$ and (u_n) is a Cauchy sequence, so it converges to $\bar{u} \in K$. Being A a monotone operator,

$$\begin{aligned}\langle Av, \bar{u} - v\rangle &= \lim_{n\to\infty} \langle Av, u_n - v\rangle \\ &\leqslant \liminf_{n\to\infty} \langle Au_n, u_n - v\rangle \leqslant 0 \quad \forall v \in K\end{aligned}$$

and, from Minty's Lemma, $\bar{u}$ is a solution for (VI).

Since every approximating sequence is convergent, it is easy to prove that there exists a unique solution for (VI). □

Now, we suppose that the variational inequality $VI(x)$ arises from a minimization problem:

$$(MP)(x)\ \inf_{u\in K} h(x,u)$$

where $h : X \times E \to R \cup \{+\infty\}$.

In Zolezzi [17] the following definition is given:

DEFINITION 2.4. Let x belong to X. The problem $(MP)(x)$ is strongly well-posed if:

(i) $\inf_{u\in k} h(x',u) > -\infty$ for all $x' \in X$;

(ii) there exists a unique $\bar{u}_x \in \operatorname{argmin} h(x,.)$;

(iii) for every sequence (x_n) converging to x and every sequence (u_n) in E such that $h(x_n,u_n) - \inf_{u\in K} h(x_n,u) \to 0$ as $n \to +\infty$, the sequence (u_n) strongly converges to $\bar{u}_x$

PROPOSITION 2.5. *For all $x \in X$ let $h(x,.)$ be a convex bounded from below and Gateaux differentiable function on K. The family of variational inequalities defined by:*

$$\begin{cases} \text{find } u \in K \text{ such that} \\ \langle h'_u(x,u), u - v\rangle \leqslant 0 \quad \forall v \in K \end{cases} \tag{4}$$

is parametrically strongly well-posed whenever the problem $(MP)(x)$ is strongly well-posed (in the sense of Definition 2.4) for every $x \in X$.

Proof. Let $x \in X$, (x_n) be a sequence converging to x and (u_n) be an approximating sequence (w.r. to (x_n)) for the problem defined by (4). Then there exists a sequence (ε_n) converging to zero such that:

$$\langle h'_u(x_n,u_n), u_n - v\rangle \leqslant \varepsilon_n \quad \forall v \in K.$$

Being $h(x_n,.)$ a convex function we have:

$$h(x_n,u_n) - h(x_n,v) \leqslant \langle h'_u(x_n,u_n), u_n - v\rangle \leqslant \varepsilon_n$$

and $h(x_n,u_n) \leqslant \inf_{v\in K} h(x_n,v) + \varepsilon_n$.

From (iii) in Definition 2.4, the sequence (u_n) has to converge to $\bar{u}_x = \operatorname{argmin} h(x,.) = T(x)$. □

Conversely we have:

PROPOSITION 2.6. *Let K be bounded and, for all $x \in X$, $h(x, .)$ be a lower semicontinuous, bounded from below and Gateaux differentiable function on K. Then the problem $(MP)(x)$ is strongly well-posed in the sense of Definition 2.4 for all $x \in X$ whenever the family defined by (4) is parametrically strongly well-posed and for all $x \in X$ argmin $h(x, .) \neq \emptyset$.*

Proof. Assume that (VI) is parametrically strongly well-posed. Let $x \in X$, (x_n) be a sequence converging to x and let (u_n) be a sequence satisfying the condition given in (iii) of Definition 2.4. That is, there exists a sequence (ε_n) decreasing to zero such that

$$h(x_n, u_n) \leqslant h(x_n, u) + \varepsilon_n \text{ for all } u \in K.$$

From Ekeland Theorem (Ekeland and Temam [5]), there exists $\bar{u}_n \in K$ such that

$$\|u_n - \bar{u}_n\| \leqslant \sqrt{\varepsilon_n}$$

and

$$\langle h'_u(x_n, \bar{u}_n), \bar{u}_n - u\rangle \leqslant \sqrt{\varepsilon_n}\|\bar{u}_n - u\| \text{ for all } u \in K.$$

Therefore $\langle h'_u(x_n, \bar{u}_n), \bar{u}_n - u\rangle \leqslant \sqrt{\varepsilon_n}\ \mathrm{diam}(K)$ for all $u \in K$ and $(\bar{u}_n)$ is an approximating sequence for the variational inequality $(VI)(x)$ (w.r. to (x_n)). The family (VI) being parametrically well-posed, $(\bar{u}_n)$ must converge to $u_x = T(x)$ and the same occurs for the sequence (u_n). In order to prove (ii) it is sufficient to consider the sequence defined by $u_n = \bar{u}_x \in$ argmin $h(x, .)$. Then, from the first part, (u_n) converges to u_x and (iii) is satisfied. □

COROLLARY 2.7. *If, for all $x \in X$, $h(x, .)$ is a convex, bounded from below and Gateaux differentiable function on K which is assumed to be also bounded, then the problem $(MP)(x)$ is strongly well-posed (in the sense of Definition 2.4) for any $x \in X$ if and only if the family (VI) is parametrically strongly well-posed.*

Now, we investigate classes of families parametrically well-posed and we start with the finite dimensional case.

PROPOSITION 2.8. *Let E be a finite dimensional space and let A be an operator on $X \times K$ such that $A(x, .)$ is monotone and hemicontinuous for all $x \in X$ and $A(., u)$ is continuous for all $u \in K$. Then (VI) is parametrically well-posed if and only if $(VI)(x)$ has a unique solution for all $x \in X$.*

Proof. Assume that $(VI)(x)$ has a unique solution u_x for all $x \in X$ and the family (VI) is not parametrically well-posed. Then there exist $x \in X$, a sequence (x_n) converging to x and an approximating sequence (v_n) (w.r. to (x_n)) which does not converge to u_x. Then one can find a sequence (ε_n) converging to zero such that: $\langle A(x_n, v_n), v_n - y\rangle \leqslant \varepsilon_n$ for all $y \in K$.

Let $\alpha_n = \frac{1}{\|v_n - u_x\|}$ and $z_n = u_x + \alpha_n(v_n - u_x)$.

Assume that the sequence (v_n) is not bounded, so that there exists a subsequence still denoted by (v_n) such that $\|v_n\| \to +\infty$. Since (z_n) is bounded and $z_n \in K$, there exists a point $z \in K$, $z \neq u_x$, towards which a subsequence $(z_k)_k$ converges.

For all $y \in K$ it results:

$$\begin{aligned}\langle A(x_k, y), z - y\rangle &= \langle A(x_k, y), z - z_k\rangle + \langle A(x_k, y), z_k - y\rangle \\ &\leqslant \|A(x_k, y)\|\|z - z_k\| + \langle A(x_k, y), u_x - y\rangle + \langle A(x_k, y), z_k - u_x\rangle \\ &= \|A(x_k, y)\|\|z - z_k\| + (1-\alpha_k)\langle A(x_k, y), u_x - y\rangle + \alpha_k\langle A(x_k, y), \\ &\quad v_k - y\rangle.\end{aligned}$$

Being $A(x_k, .)$ monotone, we have:

$$\langle A(x_k, y), v_k - y\rangle \leqslant \langle A(x_k, v_k), v_k - y\rangle \leqslant \varepsilon_k$$

and: $\langle A(x_k, y), z - y\rangle \leqslant \|A(x_k, y)\|\|z - z_k\| + \varepsilon_k\alpha_k + (1-\alpha_k)\langle A(x, u_x), u_x - y\rangle$. Being $\limsup_{k\to\infty}\langle A(x_k, u_x), u_x - y\rangle \leqslant \langle A(x, u_x), u_x - y\rangle \leqslant 0$ we obtain:

$$\langle A(x, y), z - y\rangle \leqslant \limsup_{k\to\infty}(1-\alpha_k)\langle A(x_k, u_x), u_x - y\rangle \leqslant 0 \quad \text{for all } y \in K;$$

From Minty's Lemma, the point z solves $(VI)(x)$ which contradicts the uniqueness of the solution. So (v_n) is bounded and, for some subsequence, (v_n) converges to a point v_x which has to solve the variational inequality $(VI)(x)$. So $v_x = u_x$ and this is a contradiction. □

In order to obtain a class of operators which guarantees the parametrical strong well-posedness, we consider operators A from $X \times E$ on E^* which are *strongly monotone* in the second variable, uniformly with respect to x, that is: there exists $\alpha > 0$ such that

$$\langle A(x, u) - A(x, v), u - v\rangle \geqslant \alpha\|u - v\|^2 \text{ for all } u \text{ and all } x \in X.$$

Let us recall that an operator A from E to E^* is *strongly monotone* on K if there exists $\alpha > 0$ such that

$$\langle Au - Av, u - v\rangle \geqslant \alpha\|u - v\|^2 \text{ for all } u \text{ and } v \in K.$$

PROPOSITION 2.9. *Let A be an operator strongly monotone in the second variable, uniformly with respect to x, such that, for all $x \in X$, $A(x, .)$ is hemicontinuous on a bounded convex closed subset K of E, and, for all $u \in E$, $A(\cdot, u)$ is continuous from X to (E^*, s). Then (VI) is parametrically strongly well-posed.*

Proof. Being K a bounded set, for all $x \in X$ and every sequence (x_n) converging to x, every approximating sequence (u_n) (w.r. to (x_n)) has a subsequence, still denoted by (u_n), weakly converging to $u_x = T(x)$. Then we have:

$$\alpha\|u_n - u_x\|^2 \leqslant \varepsilon_n + \langle A(x_n, u_x), u_n - u_x\rangle$$

and (u_n) is strongly convergent to u_x. □

When the operator A does not depend on the parameter x we obtain:

PROPOSITION 2.9 bis. *Assume that A is monotone and hemicontinuous on K, a closed convex subset of E. Then (VI) is strongly well-posed if there exists a point $u_0 \in K$ and a function $c : D \to [0, +\infty[$ (where D is a set of non negative numbers such that $0 \in D$) with the following properties: $c(0) = 0$ and $c(t_n) \to 0$ implies $t_n \to 0$ and such that:*

$$\langle Av, u_0 - v\rangle + c(\|u_0 - v\|) \leqslant 0 \quad \forall v \in K. \tag{5}$$

Consequently, if A is a strongly monotone and hemicontinuous operator, then (VI) is strongly well-posed.

Proof. From (5) and Minty's Lemma it follows that u_0 is a solution to (VI) and any solution $\bar{u}$ for (VI) must coincide with u_0 since:

$$0 \leqslant c(\|u_0 - \bar{u}\|) \leqslant \langle A\bar{u}, \bar{u} - u_0\rangle \leqslant 0.$$

If (u_n) is an approximating sequence for (VI) it results:

$$0 \leqslant \limsup_{n\to\infty} c(\|u_n - u_0\|) \leqslant \limsup_{n\to\infty} \langle Au_n, u_n - u_0\rangle = 0$$

and (u_n) strongly converges to u_0.

Now, let A be a strongly monotone and hemicontinuous operator and let $\alpha > 0$ such that

$$\langle Au - Av, u - v\rangle \geqslant \alpha\|u - v\|^2 \quad \text{for all } v \in E.$$

Let u_0 be the unique solution to (VI), then $\langle Av, u_0 - v\rangle + \alpha\|v - u_0\|^2 \leqslant \langle Au_0, u_0 - v\rangle \leqslant 0 \quad \forall v \in K$, so there exist the function $c(t) = \alpha t^2$ and $u_0 \in K$ satisfying (5).

3. Well-posed OPVIC

In this section we consider the problem OPVIC, presented in the introduction, with f bounded from below on $X \times K$.

First of all, let us define a concept of approximating sequence for OPVIC which generalizes the concept used in Morgan [13] for Optimization problems with constraints defined by a parametric minimum problem (also called Bilevel Programming problems or Stackelberg problems).

DEFINITION 3.1. A sequence $((x_n, u_n))$ is an approximating sequence for the problem OPVIC if:

(i) $\liminf_{n\to\infty} f(x_n, u_n) \leqslant \inf_{(x,u)\in X\times E, u\in T(x)} f(x, u)$;

(ii) $u_n \in T(x_n, \varepsilon_n)$ with $\lim_{n\to\infty} \varepsilon_n = 0$.

Then two natural concepts of well-posedness for OPVIC arise:

DEFINITION 3.2. The problem OPVIC is generalized well-posed if:
(i) $(VI)(x)$ has a unique solution for every $x \in X$;
(ii) OPVIC has at least a solution;
(iii) any approximating sequence $((x_n, u_n))$ for OPVIC has a subsequence convergent in $X \times (E, s)$ to a solution to OPVIC.

DEFINITION 3.3. The problem OPVIC is strongly well-posed if:
(i) OPVIC has a unique solution $(\bar{x}, u_{\bar{x}})$;

(ii) any approximating sequence $((x_n, u_n))$ for OPVIC converges to $(\bar{x}, u_{\bar{x}})$ in $X \times (E, s)$.

Taking into account Section 2, we are able to determine classes of problems (OPVIC) which are generalized or strongly well-posed. More precisely, we have:

THEOREM 3.4. *Assume that X is sequentially compact, f is lower semicontinuous on $X \times (E, s)$, the family (VI) is parametrically strongly well-posed and OPVIC admits at least a solution. Then the problem OPVIC is generalized well-posed.*

Proof. Let $((x_n, u_n))$ be an approximating sequence for OPVIC. Then $u_n \in T(x_n, \varepsilon_n)$ with $\lim_{n\to\infty} \varepsilon_n = 0$. Being X sequentially compact, there exists a subsequence (x_{n_k}) converging to $x_0 \in X$. Then (u_{n_k}) is an approximating sequence for the problem $(VI)(x_0)$ (with respect to (x_{n_k}) and, since (VI) is parametrically strongly well-posed, (u_{n_k}) strongly converges to $u_{x_0} = T(x_0)$, the unique solution of $(VI)(x_0)$. Moreover, from Definition 3.1, we have:

$$\liminf_{n\to\infty} f(x_n, u_n) \leqslant \inf_{(x,u)=\in X\times E, u\in T(x)} f(x, u) = \inf_{x\in X} f(x, T(x)).$$

Then:

$$f(x_0, u_{x_0}) \leqslant \liminf_{k\to\infty} f(x_{n_k}, u_{n_k}) \leqslant \inf_{x\in X} f(x, T(x))$$

and (x_0, u_{x_0}) is a solution to OPVIC. □

THEOREM 3.5. *Assume that X is sequentially compact, f is lower semicontinuous on $X \times (E, s)$, the family (VI) is parametrically strongly well- posed and OPVIC admits a unique solution $(\bar{x}, u_{\bar{x}})$. Then the problem OPVIC is strongly well-posed.*

Proof. Let $((x_n, u_n))$ be an approximating sequence for OPVIC. Argueing as in Theorem 3.1 there exists a subsequence $((x_{n_k}, u_{n_k}))$ of $((x_n, u_n))$ such that $((x_{n_k}, u_{n_k}))$ converges to $(\bar{x}, u_{\bar{x}})$. Since any converging subsequence of the approximating sequence $((x_n, u_n))$ is convergent to $(\bar{x}, u_{\bar{x}})$ in $X \times (E, s)$ it is easy to prove that the whole sequence $((x_n, u_n))$ converges to $(\bar{x}, u_{\bar{x}})$. □

Consequently, assuming that X is sequentially compact, we infer the following corollaries:

COROLLARY 3.6. *Let E be a finite dimensional space. Assume that A is an operator on $X \times K$ such that for all $x \in X$ $A(x, .)$ is monotone and hemicontinuous and, for all $u \in K$, $A(., u)$ is continuous. Let f be a lower semicontinuous real valued function on $X \times E$. Finally, assume that $(VI)(x)$ has a unique solution for all $x \in X$. Then the problem OPVIC defined by:*

$$\text{(OPVIC)} \quad \begin{cases} \text{Minimize } f(x, u) \\ \text{subject to } (x, u) \in X \times E \text{ and } u \in T(x). \end{cases}$$

is generalized well-posed. If, moreover, it admits a unique solution it is also well-posed.

COROLLARY 3.7. *Let f be a lower semicontinuous real valued function on $X \times (E, s)$. Assume that A is an operator strongly monotone in the second variable, uniformly with respect to x, such that for all $x \in X$ $A(x, .)$ is hemicontinuous on a bounded closed convex subset K of E and, for all $u \in K$, $A(., u)$ is continuous from X to (E^*, s). Then the problem OPVIC is generalized well-posed and, if it has a unique solution, it is also strongly well-posed.*

When the operator A does not depend on x we obtain:

COROLLARY 3.8. *Let f be a lower semicontinuous real valued function on $X \times (E, s)$. Assume that A is a strongly monotone and hemicontinuous operator on K or, more generically, that there exists a point $u_0 \in K$ and a function $c : D \to [0, +\infty[$ as in Proposition 2.9 bis such that (5) is satisfied. Then the problem OPVIC is generalized well-posed and, if it has a unique solution, it is also strongly well-posed.*

EXAMPLE 3.9. As pointed out in the Introduction, any method which produces approximating sequences for OPVIC allows to approach a solution. As an example we show that the sequence $((x_n, y_n))$, generated by the exact penalty method, described by Marcotte and Zhu [11], is an approximating sequence for OPVIC.

In fact, following Marcotte and Zhu, we consider the penalized problem

$$(P_\alpha) \quad \begin{cases} \text{Minimize } f(x, u) + \alpha g(x, u) \\ \text{subject to } (x, u) \in X \times E, \end{cases}$$

where g is the gap function defined in (1) and we suppose that, for all $\alpha \geqslant 0$, (P_α) has at least a solution.

Let (α_n) be an increasing sequence of positive numbers and (x_n, y_n) be a solution to the problem (P_{α_n}). From Lemma 1 in Marcotte and Zhu we have:

(1) $f(x_n, y_n) \leqslant \inf_{(x,u)\in X\times E} f(x, u) \leqslant \inf_{(x,u)\in X\times E, u\in T(x)} f(x, u)$;

(2) $g(x_{n+1}, y_{n+1}) \leqslant g(x_n, y_n)$
which imply i) and ii) of Definition 3.1 so $((x_n, y_n))$ is an approximating sequence for OPVIC.

Then, whenever OPVIC is well-posed (respectively generalized well-posed) we can deduce that the sequence $((x_n, y_n))$ converges to the solution (respectively that a subsequence of $((x_n, y_n))$ converges to a solution).

References

1. Auslender, A. (1976) Opimization, Numeriques, Masson, Paris.
2. Baiocchi, C. and Capelo, A. (1984), Variational and quasivariational in-equalities, applications to free boundary problems. New York: John Wiley and Sons.
3. Cavazzuti, E. and Morgan, J. (1978), Well-posed saddle point problem, in J.B. Hiriart-Urruty, W. Oettli and J. Stoer, (eds), *Optimization, Theory and Algorithms* (pp. 61–76). New York: Marcel Dekker.
4. Dontchev, A.L. and Zolezzi, T. (1993), Well-Posed Optimization Problems, *Lecture Notes in Mathematics 1543*. Berlin: Springer-Verlag.
5. Ekland, I. and Temam, R. (1976), Convex Analysis and variational problems, *Studies in Mathematics and Applications*. Amsterdam: North-Holland.
6. Kinderlehrer D. and Stampachia G. (1980), An introduction to variational inequality and their application. New York: Academic Press.
7. Harker, P.T. and Pang, J.S. (1990), Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and application, *Mathematical Programming* **48**: 161–220.
8. Lignola, M.B. and Morgan, J. (1994), Approximate solutions to variational inequalities and application, *Le Matematiche* **49**: 161–220.
9. Lignola, M.B. and Morgan, J. (1997), Existence of solutions to generalized bilevel programming problems, in A. Migdalas, P.M. Pardalos and P. Varbrand (eds.), *Multilevel Programming – Algorithms and Applications* (pp. 315–332). Dordrecht/Boston/London: Kluwer Academic Publishers.
10. Luo, Z.Q., Pang, J.S., Ralph, D. and Wu S.Q. (1996), Exact Penalization and Stationarity Conditions of Mathematical Programs with Equilibrium Constraints, *Mathematical Programming* **75**: 19– 76.
11. Marcotte, P. and Zhu, D.L. (1996), Exact and Inexact Penalty Methods for the Generalized Bilevel Programming Problem, *Mathematical Programming* **74**: 141–157.
12. Margiocco, M., Patrone, F. and Pusillo L. (1997), A new approach to Tikhonov well-posedness for Nash Equilibria, *Optimization* **40**: 385–400.
13. Morgan, J. (1989), Constrained well-posed two-level optimization problem, in F.H. Clarke, V.F. Demyanov and F. Giannessi (eds.), *Non-smooth optimization and related topics*, (pp. 307–325). New York: Plenum Press.
14. Outrata, J.V. (1994), On optimization problems with variational inequality constraint, *Siam J. on Optimization* **4**: 334–357.
15. Tykhonov, A.N. (1966), On the stability of the functional optimization problem, U.S.S.R. Computational Math. and Math. Phys. **6**(4): 26–33.
16. Ye, J.J., Zhu, D.L. and Zhu, Q. (1993), Generalized bilevel programming problem, DMS-646-IR, Department of Mathematics and Statistics, University of Victoria.
17. Zolezzi, T. (1995), Well-posedness criteria in optimization with application to the calculus of variation, *Nonlinear Analysis, Theory, Methods and Applications* **25**(5): 437–453.